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Applications of Canonical Transformations in Optimizing Orbital Transfers

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The application of canonic transformation theory in determining extremals for optimal transfer problems is presented. Thus, we obtain a closed-form solution for the transfer arc problem by evaluating an undefined integral representing the complete solution in the case of elliptic orbital transfers. The use of a sufficient canonicity condition allows the definition of an advantageous canonical transformation that involves determination of the generating function as a solution of the Hamilton–Jacobi equation for the elliptic and parabolic cases.

I. Introduction

EXTREMALS representing the optimal orbital transfer trajectories are solutions of a Hamiltonian system.[1] A canonical transformation will be necessary to provide a reduction of the initial canonical system.[2] Because the integration constants are canonical constants, they might be used to define a basic solution regarding the canonical approximation of the perturbation in the optimal transfer problem for a low-thrust space vehicle.

The transfer arc problem was solved by Eckenwiler[3] and Hempel.[4] Miner et al.[5] and Powers and Tapley[6] have shown how the solution can be obtained by Hamiltonian methods. The solutions provided in these references require many integrations and make the circular condition appear as a special case and not as an extension of the elliptic case.

An efficient method for the elimination of the circular singularity having the canonical constants as integration constants is given by Ref. 5. The purpose of the discussion that follows is 1) to demonstrate some of the applications of canonical transformation theory to the analysis of optimal trajectories for space vehicles and 2) to describe a solution to the elliptical coast-arc problem, which has no singularities at the circular condition.

The coast-arc solution can be used as a base solution in canonical perturbation analyses of certain classes of low-thrust problems.[7–9]

The advantage of such a Hamiltonian formulation is that once a portion of the problem is solved, e.g., the coast-arc problem defined by zero thrust, the solution of a more general problem that contains the effects of additional forces can be approached by adding terms to the Hamiltonian function instead of attacking the full problem.

II. Optimal Transfer

Let us consider the autonomous controlled system given by

$$\dot{x}_i = f_i(\boldsymbol{x}, \boldsymbol{u}) \qquad (i = 1, \dots, n) \tag{1}$$

where $\boldsymbol{x}$ is the n-dimensional state vector and $\boldsymbol{u}$ the m-dimensional control vector.

It is assumed that the functions $f_i(\boldsymbol{x}, \boldsymbol{u})$, $[\partial f_i(\boldsymbol{x}, \boldsymbol{u})/\partial x_j]$, $i, j = 1, \dots, n$, are defined and continuous on the manifold $\mathbb{R}_n \times U$, so that for the initial conditions $\boldsymbol{x}(t_0) = x^0$ and for a given control $\boldsymbol{u} = u(t)$, system (1) admits a unique solution.

If we consider the performance index

$$J = \int_{t_0}^{t_1} f_0(\boldsymbol{x}, \boldsymbol{u})\,\mathrm{d}t \tag{2}$$

the optimum problem can be stated as follows.

Given two points belonging to two different sets $x^0 \in S_0$ and $x^1 \in S_1$ and using the admissible controls $\boldsymbol{u} = \boldsymbol{u}(t)$ that transfer the system from position x^0 to position x^1, we seek to determine the control for which the performance index (2) has the lowest value.

We introduce the coordinate defined by the equation

$$\dot{x}_0 = f_0(\boldsymbol{x}, \boldsymbol{u}) \tag{3}$$

so that the system to be analyzed becomes

$$\dot{x}_i = f_i(\boldsymbol{x}, \boldsymbol{u}) \qquad (i = 0, \dots, n) \tag{4}$$

We will consider the associate system

$$\dot{\lambda}_i = -\sum_{j=0}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \qquad (i = 0, \dots, n) \tag{5}$$

which defines the vectorial function $\boldsymbol{\lambda} = [\lambda_0(t), \lambda_1(t), \dots, \lambda_n(t)]$ for the admissible control $\boldsymbol{u}(t)$, corresponding to the trajectory $\boldsymbol{x}(t)$. If the Hamiltonian function is introduced:

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}) = \sum_{i=0}^{n} \lambda_i f_i(\boldsymbol{x}, \boldsymbol{u}) \tag{6}$$

Eqs. (4) and (5) will be written in the canonic form

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 0, \dots, n) \tag{7}$$

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For $\boldsymbol{x}(t)$ and $\boldsymbol{\lambda}(t)$ that solve system (7), there exists a control $\boldsymbol{u}(t)$ for which the necessary optimum condition represented by Pontryagin's maximum principle is satisfied,

$$H[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)] = \sup_{u \in U} H[\boldsymbol{x}(t), \boldsymbol{u}, \boldsymbol{\lambda}(t)] \tag{8}$$

for each $t \in [t_0, t_1]$, where

$$\lambda_0(t) = \text{const} \leq 0 \qquad H[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)] = 0 \tag{9}$$

For nonautonomous systems,

$$\dot{x}_i = f_i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) \qquad (i = 1, \ldots, n) \tag{10}$$

and the performance index

$$J = \int_0^{t_1} f_0(\boldsymbol{x}, \boldsymbol{u}, t)\, \mathrm{d}t \tag{11}$$

the assumptions made in the autonomous case are maintained on the manifold $\mathbb{R}_{n+1} \times U$. With the substitution $t = x_{n+1}$, this case can be reduced to the case of autonomous systems.

Because $H^*(\bar{x}, \bar{\lambda})$ does not depend on the independent variable, it follows that it is a prime integral of the canonical system, i.e.,

$$H^*(\bar{x}, \bar{\lambda}) = h \tag{12}$$

where $\bar{x} = (x_1, \ldots, x_{n+1})$ and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$.

Let υ be the new independent variable defined by

$$\frac{\mathrm{d}t}{\mathrm{d}\upsilon} = \chi(\bar{x}, \bar{\lambda}) \tag{13}$$

Considering the Hamiltonian

$$\bar{H} = \chi(\bar{x}, \bar{\lambda})[H^* - h] \tag{14}$$

it will be shown that it verifies a canonical system.

Because the optimality conditions represent a Hamiltonian system described by the generalized coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$ and the generalized momenta $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, it follows that the properties of Hamiltonian systems may be used in the optimal control study.

III. Canonical Transformations

Let us consider the general form

$$\omega = \sum_{i=0}^{n} \lambda_i\, \mathrm{d}x_i - H\, \mathrm{d}t \tag{15}$$

The transformation $\{X(\boldsymbol{x}, \boldsymbol{\lambda}, t), \Lambda(\boldsymbol{x}, \boldsymbol{\lambda}, t)\}$ is canonical if there exists a function $K(X, \Lambda, t)$ such that if we denote

$$\Omega = \sum_{i=0}^{n} \Lambda_i\, \mathrm{d}X_i - K\, \mathrm{d}t \tag{16}$$

we have

$$\mathrm{d}\omega = \mathrm{d}\Omega \tag{17}$$

Condition (17) is equivalent to the condition for the existence of a function $S(\boldsymbol{x}, \boldsymbol{\lambda}, X, \Lambda, t)$ so that

$$\omega = \Omega + \mathrm{d}S \tag{18}$$

Taking into account Eqs. (15) and (16), Eq. (18) can be written as

$$\sum_{i=0}^{n} (\lambda_i\, \mathrm{d}x_i - \Lambda_i\, \mathrm{d}X_i) + (K - H)\, \mathrm{d}t = \mathrm{d}S \tag{19}$$

Because for x_i, X_i, and t the variation coincides with the differential, relation (19) may be written

$$\sum_{i=0}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) + (K - H)\delta t = \delta S + \frac{\partial S}{\partial t}\delta t \tag{20}$$

from which it follows that

$$\sum_{i=0}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) = \delta S \qquad K - H = \frac{\partial S}{\partial t} \tag{21}$$

As already shown, along the optimal trajectory, $\mathrm{d}S = 0$.

The performance index of the optimum problem stated is invariant with respect to the canonical transformation performed so that if the transformation is time independent, it follows that

$$\sum_{i=1}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) = 0 \tag{22}$$

Let us consider the transformation

$$\boldsymbol{x} = \Phi(X) \tag{23}$$

Using relation (22), it follows that

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \Phi_j}{\partial X_i} \qquad (i = 1, \ldots, n) \tag{24}$$

Relation (24) expresses the transformation of the Lagrange multipliers determined by the state-variables transformation. The transformed Hamiltonian K must satisfy Pontryagin's maximum principle along the optimal trajectory; thus, it requires the determination of the optimal control $\boldsymbol{u} = u^0(X, \Lambda)$. Taking into account the extremum condition $[\partial K(X, \Lambda)/\partial u] = 0$, it follows that on the optimal trajectory the system obtained by a canonical transformation is Hamiltonian.

Let the equations of motion of a space vehicle be linear with respect to the thrust vector $\boldsymbol{T}$.

When the thrust is off, the space vehicle's motion represents the natural trajectory for which $X_\alpha = c_\alpha\, (\alpha = 1, \ldots, p)$ are prime integrals of the motion.

Taking into account the structure of the Hamiltonian obtained by the change of the independent variable expressed by relation (14), it follows that

$$\bar{K} = \frac{\mathrm{d}t}{\mathrm{d}X_\upsilon}[K - C] \tag{25}$$

or, for the analyzed case,

$$\bar{K}_0 = \frac{\mathrm{d}t}{\mathrm{d}X_\upsilon}[K_0 - C_0] \tag{26}$$

where $C_0 = K_0$ represents the value of the Hamiltonian at the initial time of the null thrust optimal trajectory arc. The existence of the cyclic variables X_α simplifies the construction of the generating function $S(\boldsymbol{x}, \boldsymbol{\lambda}, X, \Lambda)$.

IV. Construction of the Generating Function

Let us consider the generating function

$$S(\boldsymbol{x}, \boldsymbol{\lambda}, X, \Lambda) = S_1(x_\alpha, X_\alpha) + S_2(x_k, \Lambda_k) \tag{27}$$

with the index α used for the cyclic variables.

The evaluation of the noncyclic variables' differentials is provided by

$$\Lambda_k\, \mathrm{d}X_k = \mathrm{d}(\Lambda_k, X_k) - X_k\, \mathrm{d}\Lambda_k \tag{28}$$

Denoting

$$S_2(x_k, \Lambda_k) + \sum_k (\Lambda_k X_k)_{X_k = X_k(x_k, \Lambda_k)} = \bar{S}_2(x_k, \Lambda_k) \tag{29}$$

the sufficient canonicity condition may be written, because $H = K = \text{const}$, as

$$\sum_\alpha (\lambda_\alpha\, \mathrm{d}x_\alpha - \Lambda_\alpha\, \mathrm{d}X_\alpha) + \sum_k (\lambda_k\, \mathrm{d}x_k + X_k\, \mathrm{d}\Lambda_k)$$
$$= \sum_\alpha \left(\frac{\partial S_1}{\partial x_\alpha}\mathrm{d}x_\alpha + \frac{\partial S_1}{\partial X_\alpha}\mathrm{d}X_\alpha\right) + \sum_k \left(\frac{\partial \bar{S}_2}{\partial x_k}\mathrm{d}x_k + \frac{\partial \bar{S}_2}{\partial \Lambda_k}\mathrm{d}\Lambda_k\right) \tag{30}$$

from which

$$\lambda_\alpha = \frac{\partial S_1}{\partial x_\alpha} \qquad \Lambda_\alpha = -\frac{\partial S_1}{\partial X_\alpha}$$
$$\lambda_k = \frac{\partial \bar{S}_2}{\partial x_k} \qquad X_k = \frac{\partial \bar{S}_2}{\partial \Lambda_k} \tag{31}$$

If we consider the following form for S:

$$S(\boldsymbol{x}, \boldsymbol{\lambda}, X, \Lambda) = \sum_\alpha x_\alpha X_\alpha + \sum_k x_k \Lambda_k \tag{32}$$

then

$$\boldsymbol{\lambda} = \frac{\partial S}{\partial \boldsymbol{x}} \qquad \boldsymbol{\lambda} = (\lambda_\alpha, \lambda_k) \tag{33}$$

Let X_υ be the new independent variable. The determination of the generating function S that makes the transformed Hamiltonian vanish requires us to solve the Hamilton–Jacobi partial differential equation

$$\frac{\partial S}{\partial X_\upsilon} + \bar{K}_0\left(\boldsymbol{x}, \frac{\partial S}{\partial \boldsymbol{x}}, X_\upsilon\right) = 0 \tag{34}$$

Finding the complete integral of the nonlinear equation is equivalent to the integration of the transformed canonical system.

From Eq. (34), we obtain

$$\frac{\partial S}{\partial X_\upsilon} = -\bar{K}_0 = -k = \text{const} \tag{35}$$

Integrating Eq. (35) gives

$$S = -kX_\upsilon + W(x_1, \ldots, x_n) \tag{36}$$

The unknown function W will be determined by using the separation of variables method. Thus,

$$W = \sum_i W_i(x_i) \tag{37}$$

Taking Eqs. (33) and (36) into account, we have

$$\lambda_i = \frac{\partial S}{\partial x_i} = \frac{\partial W}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{38}$$

From the prime integral of the canonical system, one can obtain

$$\lambda_i = f_i(x_i, \alpha_1, \ldots, \alpha_n, k) \tag{39}$$

so that from Eq. (38), it follows that

$$W = \sum_i \int f_i(x_i, \alpha_1, \ldots, \alpha_n, k)\,\mathrm{d}x_i \tag{40}$$

The complete integral of Eq. (34) thus will be given by

$$S = -kX_\upsilon + \sum_\alpha \alpha_\alpha x_\alpha + \sum_k f_i(x_i, \alpha_1, \ldots, \alpha_n, k)\,\mathrm{d}x_k \tag{41}$$

As stated by the Hamilton–Jacobi theorem, the functions $x_i(t)$ and $\lambda_i(t)$ determined by the system

$$\frac{\partial S}{\partial \alpha_j} = \beta_j = \text{const} \qquad \frac{\partial S}{\partial x_j} = \lambda_j \qquad (j = 1, \ldots, n) \tag{42}$$

are solutions of the canonical system.

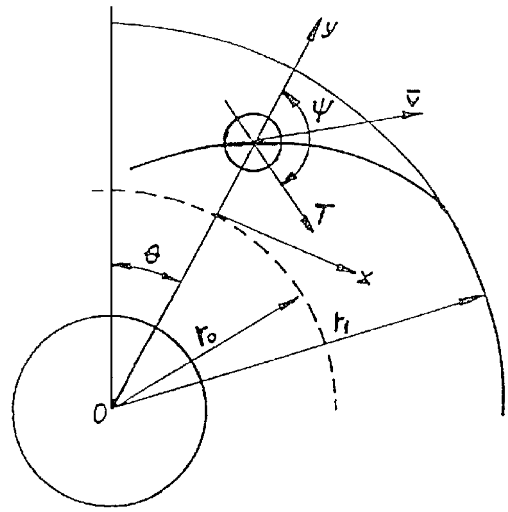

Fig. 1 Transfer between two circular orbits.

V. Optimal Orbital Transfer

Let us consider the optimal planar transfer of a space vehicle. In a polar coordinate system, if ψ is the declination angle of the thrust (as shown in Fig. 1), the equations of motion in a central field are given by

$$\dot{v}_y = \frac{v_x^2}{r_0 + y} - g\frac{r_0^2}{\left(r_0^2 + y\right)^2} - \dot{v}u_r\cos(\psi) = f_1$$
$$\dot{v}_x = -\frac{v_x v_y}{r_0 + y} - \dot{v}u_r\sin(\psi) = f_2 \tag{43}$$
$$\dot{y} = v_y = f_3 \qquad \dot{\theta} = \frac{v_x}{r_0 + y} = f_4$$

where

$$-\int_0^{t_1} \dot{v}\,\mathrm{d}t = \ln\frac{m_0}{m_1}$$

Using canonical transformation theory for the case $\dot{v} = 0$, corresponding to zero thrust, we shall determine the coast arc minimizing the time of transfer between two circular orbits. The modified Poincare variables proposed by Pinkham[10] defining an elliptic orbit are given by

$$h = \left[\frac{a(1 - e^2)}{k}\right]^{\frac{1}{2}} \qquad \theta = \theta$$
$$q = e\cos(\omega) \qquad s = e\sin(\omega) \tag{44}$$

where a is the semimajor axis, e the eccentricity, ω the argument of perihelion, h the angular momentum, and k the gravitational constant.

Let $X = \{h, q, s, \theta, t\}$ and $\boldsymbol{x} = \{v_y, v_x, y, \theta, t\}$. We will consider the transformation (23), $\boldsymbol{x} = \Phi(x)$, defined by

$$x_1 = \frac{X_2\sin(X_4) - X_3\cos(X_4)}{X_1}$$
$$x_2 = \frac{1 + X_2\cos(X_4) - X_3\sin(X_4)}{X_1}$$
$$x_3 + r_0 = \frac{X_1^2 g r_0^2}{1 + X_2\cos(X_4) + X_3\sin(X_4)} \tag{45}$$
$$x_4 = X_4 \qquad x_5 = X_5$$

By means of transformation (45) and using the inverse of Eq. (24), the Lagrange multipliers' transformation will be obtained:

$$\lambda_1 = \Lambda_2 X_1 \sin(X_4) - \Lambda_3 X_1 \cos(X_4)$$

$$\lambda_2 = \frac{X_1}{1 + X_2\cos(X_4) + X_3\sin(X_4)}\Big(X_1\Lambda_1 + \Lambda_2\{X_2 + [2 + X_2\cos(X_4) + X_3\sin(X_4)]\cos(X_4)\} + \Lambda_3\{X_3 + [2 + X_2\cos(X_4) + X_3\sin(X_4)]\sin(X_4)\}\Big)$$

$$\lambda_3 = \frac{1 + X_2\cos(X_4) + X_3\sin(X_4)}{gr_0^2 X_1} \times \left[\Lambda_1 + \Lambda_2\frac{X_3 + \cos(X_4)}{X_1} + \Lambda_3\frac{X_3 + \sin(X_4)}{X_1}\right] \tag{46}$$

$$\lambda_4 = -\Lambda_2 X_3 + \Lambda_3 X_2 + \Lambda_4 \qquad \lambda_5 = \Lambda_5$$

The Hamiltonian associated with the optimum problem may be written as

$$H^*(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}) = \sum_{i=1}^{5} \lambda_i f_i(\boldsymbol{x}, \boldsymbol{u}) \tag{47}$$

The optimality condition is given by

$$\frac{\partial H^*}{\partial \psi} = 0 \qquad \frac{\partial H^*}{\partial \psi^2} < 0 \tag{48}$$

By considering Eq. (48), the Hamiltonian (47) along the optimal trajectory can be written as

$$H^*(\boldsymbol{x}, \boldsymbol{\lambda}, u^*) = H_0(\boldsymbol{x}, \boldsymbol{\lambda}) + H_v(\boldsymbol{x}, \boldsymbol{\lambda}, u^*) \tag{49}$$

where

$$H_0(\boldsymbol{x}, \boldsymbol{\lambda}) = \lambda_1\left[\frac{x_2^2}{r_0 + x_3} - g\frac{r_0^2}{(r_0 + x_3)^2}\right] - \lambda_2\frac{x_1 x_2}{r_0 + x_3} + \lambda_3 x_1 + \lambda_4\frac{x_2}{r_0 + x_3} + \lambda_5 \tag{50}$$

$$H_v(\boldsymbol{x}, \boldsymbol{\lambda}, u^*) = -\dot{v}\sqrt{\lambda_1^2 + \lambda_2^2}\,u_r = -\dot{v}u_r(\lambda_1\cos\psi + \lambda_2\sin\psi)$$

Let $X_i = c_i$ $(i = 1, 2, 3)$ stand for Kepler's constants, i.e., the prime integrals of the motion defined by Eq. (44). The Hamiltonian obtained by the canonical transformation (45) in accordance with the null reaction force might be written as

$$K_0 = \sum_{i=1}^{5} \Lambda_i \dot{X}_i = \Lambda_4\frac{[1 + X_2\cos(X_4) + X_3\sin(X_4)]^2}{X_1^3 g r_0^2} + \Lambda_5 \tag{51}$$

We also have

$$K_v = -\dot{v}u_r X_1[A^2(X, \Lambda) + B^2(X, \Lambda)]^{\frac{1}{2}} \tag{52}$$

where the expressions of $A(X, \Lambda)$ and $B(X, \Lambda)$ are given by

$$A(X, \Lambda) = \Lambda_2\sin(X_4) - \Lambda_3\cos(X_4)$$

$$B(X, \Lambda) = \frac{1}{1 + X_2\cos(X_4) + X_3\sin(X_4)} \times \{\Lambda_1 X_1 + \Lambda_2 X_2 + \Lambda_3 X_3 + [\Lambda_2\cos(X_4) + \Lambda_3\sin(X_4)] \times [2 + X_2\cos(X_4) + X_3\sin(X_4)]\} \tag{53}$$

According to Eq. (25), we may define the Hamiltonian $\bar{K}$ with $\mathrm{d}t/\mathrm{d}X_\upsilon = 1/\dot{X}_4$,

$$\bar{K} = \bar{K}_0 + \bar{K}_v = \Lambda_4 + (\Lambda_5 - C_0) \times \frac{X_1^3 g r_0^2}{[1 + X_2\cos(X_4) + X_3\sin(X_4)]^2} - \dot{v}u_r X_1[A^2 + B^2]^{\frac{1}{2}} \times \frac{X_1^3 g r_0^2}{[1 + X_2\cos(X_4) + X_3\sin(X_4)]^2} \tag{54}$$

Developing the canonical transformation in Eq. (31), we obtain

$$\begin{gathered} p_1 = X_1 \qquad p_2 = X_2 \qquad p_3 = X_3 \qquad p_4 = \Lambda_4 \\ p_5 = \Lambda_5 - C_0 \qquad q_1 = -\Lambda_1 \qquad q_2 = -\Lambda_2 \\ q_3 = -\Lambda_3 \qquad q_4 = X_4 \qquad q_5 = X_5 \end{gathered} \tag{55}$$

The expression in Eq. (33) will be written as

$$p_i = \frac{\partial S}{\partial q_i} \tag{56}$$

or, using the new independent variable $\theta = X_4$, the expression of the Hamiltonian $\bar{K}_0$ becomes

$$\bar{K}_0 = p_4 + g r_0^2\frac{p_1^3 p_5}{[1 + p_2\cos(q_4) + p_3\sin(q_4)]^2} \tag{57}$$

The generating function S is the solution of the Hamilton–Jacobi equation (34) for $X_\upsilon = \theta$.

Because q_1, q_2, q_3, and q_5 are cyclic variables, they do not appear in the expression of $\bar{K}_0$, and it follows that p_1, p_2, p_3, and p_5 are constants.

Therefore,

$$\begin{gathered} \frac{\partial S}{\partial \theta} = -\tilde{K}_0 = -k = \alpha_0 \qquad \frac{\partial S}{\partial q_1} = p_1 = \alpha_1 = X_1 = \text{const} \\ \frac{\partial S}{\partial q_2} = p_2 = \alpha_2 = X_2 = \text{const} \\ \frac{\partial S}{\partial q_3} = p_3 = \alpha_3 = X_3 = \text{const} \qquad \frac{\partial S}{\partial q_5} = p_5 = \alpha_4 = \Lambda_5 - C \end{gathered} \tag{58}$$

From the first equation of Eq. (58), we have

$$S = -k\theta + W(q_1, \ldots, q_5) \tag{59}$$

According to the relations (37) and (38) and using Eq. (58), we get

$$W = \alpha_0\theta + \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_5 - \alpha_0 q_4 - g r_0^2 \alpha_1^3 \alpha_4 \int \frac{\mathrm{d}q_4}{[1 + \alpha_2\cos(q_4) + \alpha_3\sin(q_4)]^2} \tag{60}$$

Elliptic Case

The eccentricity $e < 1$, from which we have the discriminant

$$\Delta = 4\left(\alpha_3^2 + \alpha_2^2 - 1\right) = e^2 - 1 < 0 \tag{61}$$

In this case with the substitution $\tan(q_4/2) = \boldsymbol{u}$, the integral is transformed to

$$I = \int \frac{\mathrm{d}q_4}{[1 + \alpha_2\cos(q_4) + \alpha_3\sin(q_4)]^2} = \sum_{j=1}^{2} \int \frac{A_j\boldsymbol{u} + B_j}{\left[(1 - \alpha_2)u^2 + 2\alpha_3\boldsymbol{u} + (1 + \alpha_2)\right]^j}\,\mathrm{d}t \tag{62}$$

and by performing the analytic integrations, we get the generating function

$$\begin{aligned} S = {} & -k\theta + \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_5 - \alpha_0 q_4 - g r_0^2 \alpha_1^3 \alpha_4 \\ & \times \frac{2}{(1 - \alpha_2)\left(1 - \alpha_2^2 - \alpha_3^2\right)} \left\{ \frac{1 - \alpha_2}{\left(1 - \alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}}} \right. \\ & \times \arctan\frac{(1 - \alpha_2)\tan(q_4/2) + \alpha_3}{\left(1 - \alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}}} \\ & + \frac{1}{(1 - \alpha_2)\tan^2(q_4/2) + 2\alpha_3\tan(q_4/2) + (1 + \alpha_2)} \\ & \left. \times \left[\alpha_3 + \left(\alpha_2^2 + \alpha_3^2 - \alpha_2\right)\tan\left(\frac{q_4}{2}\right)\right] \right\} \end{aligned}$$

Parabolic Case

Because $e = 1$, we have $\Delta = 0$, and the integral I is reduced to

$$I = \frac{2}{(1-\alpha_2)^2}\sum_{j=1}^{4}\int \frac{C_j}{\{\boldsymbol{u} + [\alpha_3/(1-\alpha_2)]\}^j}\,\mathrm{d}t \qquad (63)$$

It follows that

$$S = -k\theta + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_5 - \alpha_0 q_4 + g r_0^2 \alpha_1^3 \alpha_4 \times \frac{2}{3(1-\alpha_2)[(1-\alpha_2)\tan(q_4/2)+\alpha_3]}\left[3(1-\alpha_2)^2\tan^2\left(\frac{q_4}{2}\right) - 3\alpha_3\left(1-\alpha_2^2\right)\tan\left(\frac{q_4}{2}\right) + (1-\alpha_2)^2 + \alpha_3^2(1+3\alpha_2)\right] \qquad (64)$$

VI. Elliptic Case Solution

The manifold $\{\alpha_0,\ldots,\alpha_4\}$ provides five constants of the motion. We still need 10 constants to complete the elliptic arc's representation.

The other constants will be obtained using the theorem of Jacobi, namely,

$$\beta_i = \frac{\partial S}{\partial \alpha_i} \qquad (65)$$

The unknowns are t, Λ_1, Λ_2, Λ_3, Λ_4, and $\Lambda_5 - C_0 = \alpha_4$. For an independent variable, we get $[X(\alpha,\beta), \Lambda(\alpha,\beta)]$, representing the solution of the elliptic arc problem

$$X_1 = \alpha_1 \qquad X_2 = \alpha_2 \qquad X_3 = \alpha_3 \qquad X_4 = \beta_0 + \theta$$

$$X_5 = \beta_4 + 2 g r_0^2 \alpha_1^3 \left\{ \frac{1}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}}\arctan\frac{(1-\alpha_2)\tan(\theta/2)+\alpha_3}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}} + \frac{\alpha_3 + \left(\alpha_2^2+\alpha_3^2-\alpha_2\right)\tan(\theta/2)}{(1-\alpha_2)\left(1-\alpha_2^2-\alpha_3^2\right)^2\left[(1-\alpha_2)\tan^2(\theta/2)+2\alpha_3\tan(\theta/2)+(1+\alpha_2)\right]}\right\} \qquad (66)$$

and for the adjoint variables,

$$\Lambda_1 = -\beta_1 - 6 g r_0^2 \alpha_1^2 \alpha_4 \left\{ \frac{1}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}}\arctan\frac{(1-\alpha_2)\tan(\theta/2)+\alpha_3}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}} + \frac{1}{(1-\alpha_2)\left(1-\alpha_2^2-\alpha_3^2\right)\left[(1-\alpha_2)\tan^2(\theta/2)+2\alpha_3\tan(\theta/2)+(1+\alpha_2)\right]}\left[\alpha_3 + \left(\alpha_2^2+\alpha_3^2-\alpha_2\right)\tan\left(\frac{\theta}{2}\right)\right]\right\} = F_1(\alpha_i,\beta_1)$$

$$\Lambda_2 = -\beta_2 - \frac{2 g r_0^2 \alpha_1^4 \alpha_4}{(1-\alpha_2)\left(1-\alpha_2^2-\alpha_3^2\right)\left[\tan^2(\theta/2)+2\alpha_3\tan(\theta/2)+(1+\alpha_2)\right]}\left\{3\alpha_2\left(1-\alpha_2^2-\alpha_3^2\right)^{-\frac{5}{2}}\arctan\frac{(1-\alpha_2)\tan(\theta/2)+\alpha_3}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}} + \frac{\tan^2(\theta/2)-1}{(1-\alpha_2)\tan^2(\theta/2)+2\alpha_3\tan(\theta/2)+(1+\alpha_2)}\left[\alpha_3 + \left(\alpha_2^2+\alpha_3^2-\alpha_2\right)\tan\left(\frac{\theta}{2}\right)\right] + (2\alpha_2-1)\tan\left(\frac{\theta}{2}\right) - \alpha_2\alpha_3 - \left(1-2\alpha_2^2-\alpha_3^2+\alpha_2\right)\tan\left(\frac{\theta}{2}\right) + \frac{\left(1-3\alpha_2^2-\alpha_3^2+2\alpha_2\right)}{(1-\alpha_2)\left(1-\alpha_2^2-\alpha_3^2\right)}\left[\alpha_3 + \left(\alpha_2^2+\alpha_3^2-\alpha_2\right)\tan\left(\frac{\theta}{2}\right)\right]\right\} = F_2(\alpha_i,\beta_2)$$

$$\Lambda_3 = -\beta_3 - 2 g r_0^2 \alpha_1^3 \alpha_4 \left\{3\alpha_3\left(1-\alpha_2^2-\alpha_3^2\right)^{-\frac{3}{2}}\arctan\frac{(1-\alpha_2)\tan(\theta/2)+\alpha_3}{\left(1-\alpha_2^2-\alpha_3^2\right)^{\frac{1}{2}}} + \frac{\left[2-4\alpha_2^2+5\alpha_3^2-9\alpha_2\alpha_3^2+2\left(\alpha_2^2+\alpha_3^2\right)\right]\tan^2(\theta/2)+\alpha_3\left(5-5\alpha_2^2+4\alpha_3^2\right)\tan(\theta/2)+(1+\alpha_2)\left(2-2\alpha_2^2+\alpha_3^2\right)}{(1-\alpha_2)\left(1-\alpha_2^2-\alpha_3^2\right)^2\left[(1-\alpha_2)\tan^2(\theta/2)+2\alpha_3\tan(\theta/2)+(1+\alpha_2)\right]}\right\} = F_3(\alpha_i,\beta_2)$$

$$\Lambda_5 = C_0 + \alpha_4 = F_4(\alpha_4)$$

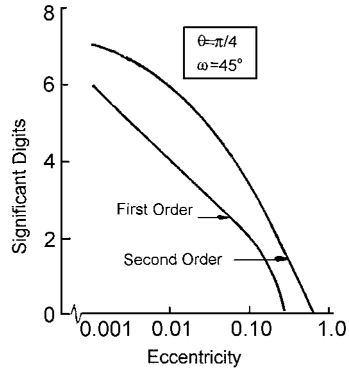

Fig. 2 Validity of the Y approximation for a small range angle.

VII. Numerical Results

The solution form (66) is especially useful for near-circular coast arcs because then $q = X_2 = \alpha_2$ and $s = X_3 = \alpha_3$ are near zero. Also the arc-tangent function Y can be approximated to second order in eccentricity by

$$Y \approx \tfrac{1}{2}\left[\theta - X_2\sin\theta + X_3(1+\cos\theta) + X_2 X_3\sin^2\theta + \tfrac{1}{2}\left(X_2^2 - X_3^2\right)\sin\theta\cos\theta\right] \qquad (67)$$

Equation (67) is extremely useful in the study of certain multirevolution optimal low-thrust trajectories because then the troublesome

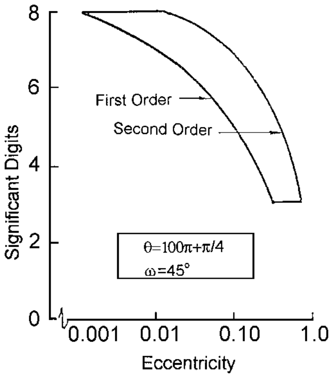

Fig. 3 Validity of the *Y* approximation for a large angle (after 50 revolutions).

task of keeping track of the arctangent function is avoided.[7,9] For example, on a representative optimal escape trajectory (70.5 revolutions and travel time $= 1.28 \times 10^6$ s) with circular initial conditions, the approximation of Eq. (67) held at least six-digit accuracy for the first 85% of the trajectory and at least three-digit accuracy for the remainder of the trajectory. On representative multirevolution circular orbit transfers, the approximation held six-digit accuracy for the entire trajectory. The reason for this convergence behavior is indicated by Figs. 2 and 3. That is, for the major initial portion of both classes of trajectories, the eccentricity is less than 0.01.

VIII. Conclusions

The paper applies the theory of canonical transformations to the study of optimal orbit transfer. The modified Poincare variables are used in this context. The generating function is determined as a complete integral of the Hamilton–Jacobi equation. A procedure for obtaining a set of canonical constants, which defines the elliptic coast arc, is described.

As an application of the theory, we consider the minimization of the orbital transfer time in a central force field. One of the nonsingular arcs was analyzed.

A natural extension of this investigation would be to start with the Hamiltonian $K[X(\alpha, \beta), \Lambda(\alpha, \beta)]$ and attempt either to canonically transform the system or to perform a canonical perturbation analysis to incorporate the effects of the engine thrust. Inasmuch as the time rates of changes of α_i and β_i will be multiplied by the thrust, they should be slowly varying elements for very low-thrust missions.

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